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A Counterexample to Strong Unicity in Monotone Approximation

YATES FLETCHER AND JOHN A. ROULIER*

Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27607

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1. INTRODUCTION

Let Π_n denote the set of polynomials of degree n or less and let $\|\ \|$ be the sup norm on $C[a, b]$. It is well known that

$$\begin{aligned}&\text{for each } f \in C[a, b] \text{ there exists a unique } p_f \in \Pi_n\\ &\text{which satisfies } \|f - p_f\| \leqslant \|f - p\| \quad \forall p \in \Pi_n .\end{aligned} \tag{1.1}$$

For fixed n, p_f is called the polynomial of best approximation to f. One of the basic theorems strengthening this result is the Strong Unicity Theorem which guarantees the existence of a positive constant γ depending only on f for which the inequality

$$\|f - p\| \geqslant \|f - p_f\| + \gamma \|p - p_f\| \qquad \text{holds } \forall p \in \Pi_n . \tag{1.2}$$

See, for example, Cheney [1, pp. 80–81]. We say in this case that p_f is *strongly unique*.

In the theory of monotone approximation the set of approximating elements Π_n is replaced by the set $M_n = \{ p \in \Pi_n \mid p'(x) \geqslant 0 \ \forall x \in [a, b]\}$. Lorentz and Zeller [4] have shown that (1.1) holds if we replace Π_n by M_n (p_f is then called the monotone polynomial of best approximation). Our main result is an example which shows that (1.2) need not hold with Π_n replaced by M_n and p_f replaced by the best approximation to f from M_n.

Let $f \in C[a, b]$ and let $p_f \in M_n$ be the monotone polynomial of best approximation to f. We define the two sets of "extreme points" in $[a, b]$

$$A = \{x \mid |f(x) - p_f(x)| = \|f - p_f\|\} \tag{1.3}$$

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and

$$B = \{x \mid p_f(x) = 0\}. \tag{1.4}$$

Also define $\sigma(x) = [f(x) - p_f(x)]/\|f - p_f\|$ for $f \notin M_n$. Lorentz and Zeller [4] prove the following:

LEMMA 1.1. *p_f is the monotone polynomial of best approximation to f from M_n if and only if there exist points*

$$x_i \in A, \qquad i = 1, 2, \ldots, \mu$$

and

$$y_j \in B, \qquad j = 1, 2, \ldots, \lambda$$

and corresponding numbers $\alpha_i > 0$, $i = 1, 2, \ldots, \mu$ and $\beta_j > 0$, $j = 1, 2, \ldots, \lambda$ such that $\mu + \lambda \leqslant n + 2$ and

$$\sum_{i=1}^{\mu} \alpha_i \sigma(x_i)\, p(x_i) + \sum_{j=1}^{\lambda} \beta_j p'(y_j) = 0 \tag{1.5}$$

for all $p \in \Pi_n$.

Moreover, if we let e denote the number of the y_j which are equal to a or b, the proof of Theorem 9 of [4] gives

$$\mu + 2\lambda - e \geqslant n + 2. \tag{1.6}$$

The following theorem follows from the above result, but has a direct proof and is due to Roulier [6].

LEMMA 1.2. *If p_f is the best approximation to f from M_n and if $B = \varnothing$ (i.e., $p_f'(x) > 0$ on $[a, b]$) then in fact p_f is the best approximation to f from Π_n on $[a, b]$.*

These two results together with the results on Birkhoff interpolation used in [4] will be our chief tools in the remaining sections.

The study of strong unicity and Lipschitz constants in settings other than the classical one have been studied in [3] for shrinking intervals and in [2] and [8] for changing dimension.

The last two sections obtain modified strong unicity and continuity results for the best monotone approximation.

2. A Counterexample

Example of a function whose monotone polynomial of best approximation is not strongly unique. Let

$$f(x) = \frac{1}{2} - x^2 + \left(x - \frac{1}{3^{1/2}}\right)^3, \qquad x \in [-1, 1].$$

Claim I. *The best monotone approximation to f out of Π_3 on $[-1, 1]$ is $p_f(x) = (x - 1/3^{1/2})^3$.*

Proof. For the proof of this claim we appeal to Lemma 1.1. We see that

$$A = \{x \mid |f(x) - p_f(x)| = \|f - p_f\|\} = \{-1, 0, 1\},$$

$$B = \{x \mid p_f'(x) = 0\} = \left\{\frac{1}{3^{1/2}}\right\},$$

$$\sigma(x) = \frac{f(x) - p_f(x)}{\|f - p_f\|}, \qquad x \in A \Rightarrow \sigma(-1) = -1,$$

$$\sigma(0) = 1, \quad \text{and} \quad \sigma(1) = -1.$$

The fact that

$$(2 - 3^{1/2})[\sigma(-1)\,p(-1)] + 4[\sigma(0)\,p(0)] + (2 + 3^{1/2})[\sigma(1)\,p(1)] + 2(3)^{1/2}\,p'\left(\frac{1}{3^{1/2}}\right) = 0 \tag{2.1}$$

holds for all $p \in \Pi_3$ allows us to invoke Lemma 1.1 which establishes the claim. In order to verify (2.1), simply observe that it is valid for $p(x) = 1$, $p(x) = x$, $p(x) = x^2$, and $p(x) = x^3$.

Now fix $\alpha \in (0, 1)$ and define:

$$p_\alpha(x) = \left(x - \frac{1}{3^{1/2}}\right)^3 + \alpha x(x^2 - (1 - \alpha))$$
$$= (1 + \alpha)\,x^3 - 3^{1/2}x^2 + (1 - \alpha + \alpha^2)x - \frac{1}{3(3)^{1/2}}.$$

Then $p_\alpha'(x) = 3(1 + \alpha)\,x^2 - 2(3)^{1/2}x + (1 - \alpha + \alpha^2)$. The discriminant of p_α' is $12 - 12[(1 + \alpha)(1 - \alpha + \alpha^2)] = -12\alpha^3 < 0$. Therefore p_α' does not change sign and since $p_\alpha'(0) > 0$ we have $p_\alpha'(x) > 0$, $x \in [-1, 1]$. Thus, $p_\alpha \in M_3$.

Claim II. $\|p_\alpha - p_f\| = [2\alpha/3(3)^{1/2}](1 - \alpha)^{3/2}$ *for α sufficiently small.*

Proof. Note that $[p_\alpha - p_f](x) = \alpha x^3 - \alpha(1 - \alpha)x$. It is a simple exercise to show that for α sufficiently small $|p_\alpha(x) - p_f(x)| = \|p_\alpha - p_f\|$ at $x = \pm((1 - \alpha)/3)^{1/2}$, and that $\|p_\alpha - p_f\| = [2\alpha/3(3)^{1/2}](1 - \alpha)^{3/2}$ in this case.

CLAIM III. $\|f - p_\alpha\| = \frac{1}{2} + \alpha^2$.

Proof. Observe that $|[f - p_\alpha](1)| = |-\frac{1}{2} + \alpha - \alpha^2 - \alpha| = \frac{1}{2} + \alpha^2$. Thus we must show that

$$|[f - p_\alpha](x)| \leqslant \tfrac{1}{2} + \alpha^2, \qquad x \in [-1, 1]. \tag{2.2}$$

So, suppose $|x| \leqslant 1$. Then we have,

$$x^2(1 + x) \leqslant 4 \tag{2.3}$$

and

$$(1 - x^2)(1 - x) \leqslant 4. \tag{2.4}$$

From (2.3) we obtain $x^2(1 - x)^2(1 + x)^2 \leqslant 4(1 - x)^2(1 + x)$. Thus $(x - x^3)^2 - 4(1 - x)(1 - x^2) \leqslant 0$, and the quadratic in α $(1 - x)\,\alpha^2 + (x - x^3)\alpha + (1 - x^2)$ does not change sign (and thus is nonnegative). That is, $0 \leqslant \alpha^2 - \alpha^2 x + \alpha x - \alpha x^3 + 1 - x^2$, which gives

$$-1 - \alpha^2 \leqslant -x^2 + (x - x^3)\alpha - x\alpha^2. \tag{2.5}$$

From (2.4) we obtain $(1 + x)(1 - x)^2 \leqslant 4$. Thus $x^2(1 + x)^2(1 - x)^2 \leqslant 4(1 + x)\,x^2$, and $(x - x^3)^2 - 4(1 + x)\,x^2 \leqslant 0$. Hence, the quadratic in α, $(1 + x)\,\alpha^2 - (x - x^3)\alpha + x^2$ does not change sign (and is thus nonnegative.) That is, $0 \leqslant \alpha^2 + \alpha^2 x - \alpha x + \alpha x^3 + x^2$ which gives

$$-x^2 + (x - x^3)\alpha - x\alpha^2 \leqslant \alpha^2. \tag{2.6}$$

It now follows from (2.5) and (2.6) that $-1 - \alpha^2 \leqslant -x^2 + (x - x^3)\alpha - x\alpha^2 \leqslant \alpha^2$. Hence, $-\frac{1}{2} - \alpha^2 \leqslant \frac{1}{2} - x^2 + (x - x^3)\alpha - x\alpha^2 \leqslant \frac{1}{2} + \alpha^2$, and so $|[f - p_\alpha](x)| \leqslant \frac{1}{2} + \alpha^2$. We now combine Claims I, II, and III to find that for $0 < \alpha < 1$ and α sufficiently small,

$$\frac{\|f - p_\alpha\| - \|f - p_f\|}{\|p_\alpha - p_f\|} = \frac{\frac{1}{2} + \alpha^2 - \frac{1}{2}}{\dfrac{2\alpha}{3(3)^{1/2}}(1 - \alpha)^{3/2}} = \frac{3(3)^{1/2}}{2}\,\alpha(1 - \alpha)^{-3/2}.$$

Hence we can make the above expression as small as desired by taking α sufficiently small. This clearly shows the impossibility of obtaining a "strong unicity" constant for f.

3. STRONG UNICITY FOR SOME CASES

In this section we show that the monotone polynomial of best approximation is strongly unique if it's degree is less than or equal to two.

THEOREM 3.1. *If p_f is the best approximation from M_n to f on $[a, b]$ and if the degree of p_f is 0, 1, or 2 then p_f is strongly unique.*

This shows that the counterexample in the previous section could not have been of lower degree.

LEMMA 3.1. *If $p_f \in M_n$ and $\{x_i\}_{i=1}^{\mu} \subset A$ are as in Lemma 1.1 and if $p_f - p \in M_n$ and $\max_{1 \leqslant i \leqslant \mu} \sigma(x_i)\, p(x_i) \leqslant 0$, then $p \equiv 0$.*

Proof. Suppose $p_f - p \in M_n$ and $\max_{1 \leqslant i \leqslant \mu} \sigma(x_i)\, p(x_i) \leqslant 0$. Then $\sum_{i=1}^{\mu} \alpha_i \sigma(x_i)\, p(x_i) \leqslant 0$ (since $\alpha_i > 0$), and we conclude from (1.5) that $\sum_{j=1}^{\lambda} \beta_j\, p'(y_j) \geqslant 0$. But $p_f \in M_n$ implies that $p_f'(y_j) = 0$, $j = 1,\ldots, \lambda$, and $p_f''(y_j) = 0$ for $a < y_j < b$. Thus $p'(y_j) \leqslant 0, j = 1,\ldots, \lambda$, and (1.5) gives

$$p(x_i) = 0, \qquad i = 1,\ldots, \mu, \tag{3.1}$$

$$p'(y_j) = 0, \qquad j = 1,\ldots, \lambda. \tag{3.2}$$

Therefore $[p_f - p]'(y_j) = 0$, $j = 1,\ldots, \lambda$. But $p_f - p \in M_n$ implies that $[p_f - p]''(y_j) = 0$ if $a < y_j < b$. Thus

$$p''(y_j) = 0 \qquad \text{for} \quad a < y_j < b. \tag{3.3}$$

Hence, (3.1), (3.2), and (3.3) furnish the data for a Birkhoff interpolation problem. Now, (1.6) and the techniques in [4] prove the lemma.

LEMMA 3.2. *If $\lambda = 1$, $y_1 = a$ (or b), and if $\{p_k\}_{k=1}^{\infty}$ satisfies*

$$p_k \in \Pi_n\,, \tag{3.4}$$

$$\lim_{k\to\infty} p_k = \tilde{p} \in \Pi_n \qquad \textit{uniformly on } [a, b], \tag{3.5}$$

$$p_k'(y_1) \leqslant 0 \qquad \textit{for} \quad k = 1, 2,\ldots, \tag{3.6}$$

$$\limsup_{k\to\infty} \left(\max_{1 \leqslant i \leqslant \mu} \sigma(x_i)\, p_k(x_i) \right) \leqslant 0 \tag{3.7}$$

then $\tilde{p} \equiv 0$.

Proof. Lemma 1.1 gives constants

$$\alpha_i > 0, \qquad i = 1,\ldots, \mu \quad \text{and} \quad \beta_1 > 0$$

for which

$$\sum_{i=1}^{\mu} \alpha_i \sigma(x_i)\, p(x_i) + \beta_1 p'(y_1) = 0$$

for all $p \in \Pi_n$.

Thus for each $k = 1, 2, \ldots$ we have

$$\sum_{i=1}^{\mu} \alpha_i \sigma(x_i)\, p_k(x_i) = -\beta_1 p_k'(y_1) \geqslant 0.$$

Moreover, since $p_k' \to \tilde{p}'$ uniformly on $[a, b]$ also, we have $\tilde{p}'(y_1) \leqslant 0$. Thus, as above, we have

$$\sum_{i=1}^{\mu} \alpha_i \sigma(x_i)\, \tilde{p}(x_i) \geqslant 0. \tag{3.8}$$

On the other hand, it follows from (3.5) and (3.7) that

$$\max_{1 \leqslant i \leqslant \mu} \sigma(x_i)\, \tilde{p}(x_i) \leqslant 0. \tag{3.9}$$

It now follows from (3.8) and (3.9) and the fact that $\alpha_i > 0$ for $i = 1, 2, \ldots, \mu$, that $\sigma(x_i)\, \tilde{p}(x_i) = 0$ for $i = 1, 2, \ldots, \mu$. But $\sigma(x_i) = \pm 1$ for $i = 1, \ldots, \mu$. Hence,

$$\tilde{p}(x_i) = 0 \qquad \text{for} \quad i = 1, 2, \ldots, \mu. \tag{3.10}$$

Now, it follows from (1.5) and the fact that $\lambda = 1$ and y_1 is an endpoint, that,

$$\mu \geqslant n + 1. \tag{3.11}$$

Hence, it follows from (3.10) and (3.11) that $\tilde{p}$ has at least $n + 1$ zeros. Thus, the proof of Lemma 3.2 is complete.

Define,

$$Q = \left\{ q \mid q(x) = \frac{p(x)}{\| p \|},\, p_f - p \in M_n,\, \| p \| \neq 0 \right\}. \tag{3.12}$$

LEMMA 3.3. *If* $\lambda = 1$, *and* $y_1 = a$ *(or* b*) then* $\inf_{q \in Q} \max_{1 \leqslant i \leqslant \mu} \sigma(x_i)\, q(x_i) = \gamma > 0$.

Proof. Assume that the lemma is false. Then there is a sequence $\{q_k\}_{k=1}^{\infty} \subset Q$ such that

$$\limsup_{k \to \infty} \max_{1 \leqslant i \leqslant \mu} \sigma(x_i)\, q_k(x_i) \leqslant 0.$$

Since, $\| q_k \| = 1$ for $k = 1, 2, \ldots$, we may assume without loss of generality that $\lim_{k \to \infty} q_k = q \in \Pi_n$ and the convergence is uniform. Moreover, $\| q \| = 1$ and

$$\max_{1 \leqslant i \leqslant \mu} \sigma(x_i)\, q(x_i) \leqslant 0.$$

On the other hand

$$q_k'(y_1) = \frac{p_k'(y_1)}{\| p_k \|} \leqslant \frac{p_f'(y_1)}{\| p_k \|} = 0.$$

By Lemma 3.2 then, we see that $q \equiv 0$. But this contradicts $\| q \| = 1$. Thus, Lemma 3.3 is proven.

Proof of Theorem 3.1. The proof considers three cases:

Case 1. $p_f'(x) > 0$ on $[a, b]$. In this case Lemma 1.2 shows that p_f is the ordinary best approximation to f from Π_n. Thus strong unicity follows from the classical strong unicity theorems.

Case 2. $p_f'(x) \equiv 0$ on $[a, b]$. Define Q as in (3.12). We see that if $q \in Q$ then

$$q'(x) = \frac{p'(x)}{\| p \|} \leqslant \frac{p_f'(x)}{\| p \|} = 0.$$

Now assume that

$$\inf_{q \in Q} \max_{1 \leqslant i \leqslant \mu} \sigma(x_i)\, q(x_i) \leqslant 0.$$

Then there is a sequence $\{q_n\}_{n=1}^{\infty}$, $q_n \in Q$ for $n = 1, 2, \ldots$, such that

$$\limsup_{n \to \infty} \max_{1 \leqslant i \leqslant \mu} \sigma(x_i)\, q_n(x_i) \leqslant 0.$$

Since $\{q_n\}_{n=1}^{\infty}$ has a convergent subsequence we may without loss of generality assume $\lim_{n \to \infty} q_n = \tilde{q}$ uniformly on $[a, b]$. Now $\| \tilde{q} \| = 1$ and

$$\max_{1 \leqslant i \leqslant \mu} \sigma(x_i)\, \tilde{q}(x_i) \leqslant 0.$$

Moreover, since $q_n'(x) \leqslant 0$ on $[a, b]$ for all n we have

$$\tilde{q}'(x) \leqslant 0 \qquad \text{on } [a, b],$$

implying that $p_f - \tilde{q} \in M_n$. Thus Lemma 3.1 gives $\tilde{q} \equiv 0$, a contradiction. Hence,

$$\inf_{q \in Q} \max_{1 \leqslant i \leqslant \mu} \sigma(x_i)\, q(x_i) = \tau > 0.$$

To show strong unicity now let $p \in M_n$, $p \not\equiv p_f$, and define

$$q(x) = \frac{p_f(x) - p(x)}{\| p_f - p \|}.$$

Then $q \in Q$ and

$$\max_{1 \leqslant i \leqslant \mu} \sigma(x_i)\, q(x_i) \geqslant \tau > 0$$

with τ independent of p.

Choose $x^* \in A$ such that

$$\sigma(x^*)\, q(x^*) \geqslant \tau.$$

Now,

$$\begin{aligned} \|f - p\| &\geqslant \sigma(x^*)(f(x^*) - p(x^*)) \\ &= \sigma(x^*)(f(x^*) - p_f(x^*)) + \sigma(x^*)(p_f(x^*) - p(x^*)) \\ &= \|f - p_f\| + \sigma(x^*)\, q(x^*)\, \|p_f - p\| \\ &\geqslant \|f - p_f\| + \tau\, \|p_f - p\|. \end{aligned}$$

This completes the proof for Case 2.

Case 3. Either $p_f'(a) = 0$ or $p_f'(b) = 0$.

We assume without loss of generality that $p_f'(a) = 0$. In this case Lemma 3.3 applies. Thus if Q is defined as in the previous case we have

$$\inf_{q \in Q} \max_{1 \leqslant i \leqslant \mu} \sigma(x_i)\, q(x_i) = \tau > 0.$$

The remainder of the proof now proceeds as in the last part of the previous case. ∎

4. Modified Strong Unicity

In this section we present two theorems which show that strong uniqueness results of a modified nature are possible for all n. The first result gives (1.2) for all n but only for all p satisfying $0 \leqslant p'(x) \leqslant p_f'(x)$ on $[a, b]$. The second result holds true for all $p \in M_n$ but $\|p - p_f\|$ in (1.2) is replaced by $\|p - p_f\|'$, where $\|\cdot\|'$ is a certain seminorm.

Theorem 4.1. *Let $f \in C[a, b]$ and let p_f be the monotone polynomial of best approximation to f on $[a, b]$. Then there is a number $\tau > 0$ such that*

$$\|f - p\| \geqslant \|f - p_f\| + \tau\, \|p_f - p\|$$

for all p for which both $p \in M_n$ and $p'(y_j) = 0$ for $j = 1,\ldots, \lambda$. (This includes the case $p \in M_n$ and $p_f - p \in M_n$.)

Proof. Consider the set $Q_1 = \{q \mid q(x) = p(x)/\|p\|$, where $p'(y_j) = 0$, $j = 1,..., \lambda$ and $p_f - p \in M_n$, and where $p \neq 0\}$.

We will show that

$$\inf_{q \in Q_1} \max_{x \in A} \sigma(x)\, q(x) = \tau > 0, \tag{4.1}$$

where σ is as defined in Section 1. To see this, assume that (4.1) is false. Then there exist $q_m \in Q_1$, $m = 1, 2,...,$ such that

$$\limsup_{m \to \infty} \max_{x \in A} \sigma(x)\, q_m(x) \leqslant 0.$$

Moreover, we may without loss of generality assume that there is q such that

$$\lim_{m \to \infty} q_m = q$$

uniformly on $[a, b]$. Now,

$$q \in \Pi_n$$

and so by (1.5) we have

$$\sum_{i=1}^{\mu} \alpha_i \sigma(x_i)\, q(x_i) + \sum_{j=1}^{\lambda} \beta_j q'(y_j) = 0. \tag{4.2}$$

Furthermore, we have

$$q_m'(y_j) = 0 \qquad \text{for} \quad j = 1, 2,..., \lambda,$$

and

$$q_m''(y_j) = 0 \qquad \text{for each } y_j \in (a, b). \tag{4.3}$$

Thus, q must also satisfy

$$q'(y_j) = 0 \qquad \text{for} \quad j = 1, 2,..., \lambda,$$

and

$$q''(y_j) = 0 \qquad \text{for all} \quad y_j \in (a, b).$$

This, the fact that $\sigma(x)\, q(x) \leqslant 0$ for all x in A, and (4.2) show that

$$q(x_i) = 0 \qquad \text{for} \quad i = 1,..., \mu. \tag{4.4}$$

Hence, by the same method as that used in the proof of Lemma 3.1 we have $q \equiv 0$. But $\| q_m \| = 1$ for $m = 1, 2,...$. Hence, $\| q \| = 1$. This is a contradiction. This proves (4.1). Theorem 4.1 now follows easily as we will show.

Let $p \in \Pi_n$ satisfy $p \in M_n$ and $p'(y_j) = 0$ for $j = 1,..., \lambda$. Let $r(x) = p_f(x) - p(x)$. If $p \neq p_f$ then $\| r \| \neq 0$ and

$$q(x) = \frac{r(x)}{\| r \|} \in Q_1 .$$

Now, by (4.1)

$$\max_{x \in A} \sigma(x)\, q(x) \geqslant \tau > 0.$$

Choose $\bar{x} \in A$ such that

$$\sigma(\bar{x})\, q(\bar{x}) \geqslant \tau.$$

Then,

$$\sigma(\bar{x})(p_f(\bar{x}) - p(\bar{x})) \geqslant \tau \| p_f - p \|.$$

Now observe that,

$$\begin{aligned} \| f - p \| &\geqslant \sigma(\bar{x})(f(\bar{x}) - p(\bar{x})) = \sigma(\bar{x})(f(\bar{x}) - p_f(\bar{x})) + \sigma(\bar{x})(p_f(\bar{x}) - p(\bar{x})) \\ &= \| f - p_f \| + \sigma(\bar{x})(p_f(\bar{x}) - p(\bar{x})) \geqslant \| f - p_f \| + \tau \| p_f - p \|. \quad \blacksquare \end{aligned}$$

THEOREM 4.2. *Let the hypotheses be those of Theorem* 4.1. *Then there is a number* $\rho > 0$ *such that*

$$\| f - p \| \geqslant \| f - p_f \| + \rho \| p_f - p \|'$$

for all $p \in M_n$, *where*

$$\| g \|' = \max_{\substack{1 \leqslant i \leqslant \mu \\ 1 \leqslant j \leqslant \lambda}} (| g(x_i)|, | g'(y_j)|)$$

and where x_i , $i = 1,..., \mu$ *and* y_j , $j = 1,..., \lambda$ *are as in Lemma* 1.1.

Proof. For $p \in \Pi_n$ define

$$\| p \|' = \max_{\substack{1 \leqslant i \leqslant \mu \\ 1 \leqslant j \leqslant \lambda}} (| p(x_i)|, | p'(y_j)|).$$

$\| \; \|'$ is easily seen to be a seminorm. Now define

$$Q(\mu, \lambda) = \left\{ q \mid q(x) = \frac{p(x)}{\| p \|'} , \text{ where } \| p \|' \neq 0, \text{ and } p_f - p \in M_n \right\}.$$

We claim that,

$$\inf_{q \in Q(\mu,\lambda)} \max_{1 \leqslant i \leqslant \mu} \sigma(x_i)\, q(x_i) = \rho > 0.$$

To see this, assume that there are

$$q_m \in Q(\mu, \lambda), \qquad m = 1, 2, \ldots,$$

such that

$$\limsup_{m\to\infty} \max_{1\leqslant i\leqslant\mu} \sigma(x_i)\, q_m(x_i) \leqslant 0. \tag{4.5}$$

Also, since $p_f - p \in M_n$,

$$q_m'(y_j) \leqslant 0 \qquad \text{for} \quad j = 1,\ldots, \lambda.$$

Furthermore, it follows from (1.5) and this last expression, that

$$\max_{1\leqslant i\leqslant\mu} \sigma(x_i)\, q_m(x_i) \geqslant 0.$$

Thus, with (4.5), we have

$$\lim_{m\to\infty} \max_{1\leqslant i\leqslant\mu} \sigma(x_i)\, q_m(x_i) = 0.$$

But then by (1.5) and the fact that $q_m'(y_j) \leqslant 0$ for $j = 1,\ldots, \lambda$ we have

$$\lim_{m\to\infty} q_m'(y_j) = 0 \qquad \text{for} \quad j = 1,\ldots, \lambda.$$

Thus again by (1.5) we get $\lim_{m\to\infty} \sigma(x_i)\, q_m(x_i) = 0$ for $i = 1,\ldots, \mu$. Since $\sigma(x_i) = \pm 1$ for $i = 1,\ldots, \mu$, and $\|\cdot\|'$ is a continuous seminorm on Π_n we have

$$\lim_{m\to\infty} \| q_m \|' = 0.$$

But for each m, $\| q_m \|' = 1$. This is a contradiction. Thus, the claim is proved. The remainder of the proof proceeds as in the last part of the proof of Theorem 4.1. ∎

5. Continuity of the Operator $T(f) = p_f$

It is well known that the strong unicity theorem in the classical case implies a local Lipschitz condition for the best approximation operator. This, of course, implies the continuity of this operator. See Cheney [1, p. 82].

In this section, we will obtain a modified Lipschitz condition for the best monotone approximation operator and then use this to conclude that this operator is continuous.

For each f in $C[a, b]$, as above, let p_f denote the best approximation to f from M_n.

THEOREM 5.1. *Let $f \in C[a, b]$. There exists a positive number K such that for all $g \in C[a, b]$*

$$\| p_f - p_g \|' \leqslant K \| f - g \|. \tag{5.1}$$

$\| \ \|'$ is as defined in Theorem 4.2, and we may take $K = 2/\rho$, where ρ is the constant obtained in Theorem 4.2.

Proof. The proof proceeds exactly as in the classical case [1, p. 82]. Since the proof is short, we reproduce it here. Observe that by Theorem 4.2

$$\| p_f - p \|' \leqslant \frac{1}{\rho} (\| f - p \| - \| f - p_f \|) \tag{5.2}$$

for any $p \in M_n$.

Thus if $p = p_g$ for some $g \in C[a, b]$ we have from (5.2)

$$\begin{aligned} \| p_f - p_g \|' &\leqslant \frac{1}{\rho} (\| f - p_g \| - \| f - p_f \|) \\ &\leqslant \frac{1}{\rho} (\| f - g \| + \| g - p_g \| - \| f - p_f \|) \\ &\leqslant \frac{1}{\rho} (\| f - g \| + \| g - p_f \| - \| f - p_f \|) \\ &\leqslant \frac{1}{\rho} (\| f - g \| + \| g - f \| + \| f - p_f \| - \| f - p_f \|) \\ &\leqslant \frac{2}{\rho} \| f - g \|. \quad \blacksquare \end{aligned}$$

The second theorem can now be proved. The proof depends on (5.1) and the theory of Birkhoff interpolation.

THEOREM 5.2. *The operator $T(f) = p_f$ is continuous on $C[a, b]$.*

Proof. It suffices to show that if $f \in C[a, b]$ and if $\{g_m\}_{m=1}^{\infty}$ is a sequence of elements of $C[a, b]$ satisfying $\lim_{m \to \infty} g_m = f$ uniformly on $[a, b]$. Then $\lim_{m \to \infty} T(g_m) = T(f)$ uniformly on $[a, b]$. Consider such a sequence $\{g_m\}_{m=1}^{\infty}$.

It follows immediately from (5.1) that

$$\lim_{m \to \infty} \| T(f) - T(g_m) \|' = 0. \tag{5.3}$$

Furthermore,

$$\begin{aligned}\| T(g_m)\| &\leqslant \| T(g_m) - g_m \| + \| g_m \| \\ &\leqslant \| g_m \| + \| g_m \| \leqslant 1 + 2\|f\|\end{aligned}$$

for m sufficiently large. Thus, $\{T(g_m)\}_{m=1}^{\infty}$ is bounded. Now assume that $\lim_{m\to\infty} T(g_m) \neq T(f)$. Then there is $\epsilon_0 > 0$ and a subsequence $\{T(g_{m_k})\}_{k=1}^{\infty}$ such that

$$\| T(g_{m_k}) - T(f)\| \geqslant \epsilon_0 \tag{5.4}$$

$k = 1, 2,\ldots$. Furthermore $\{T(g_{m_k})\}_{k=1}^{\infty}$ is bounded. Hence, this sequence has a subsequence which converges. We may assume without loss of generality that the sequence itself converges to $q \in M_n$.

We will now show that $q = T(f)$, and thus reach a contradiction to the above assumption.

Define $p_k = T(g_{m_k})$ and $p_f = T(f)$. It follows from (5.3) that

$$\lim_{k\to\infty} p_k(x_i) = p_f(x_i), \qquad i = 1,\ldots, \mu$$

and

$$\lim_{k\to\infty} p_k'(y_j) = p_f'(y_j), \qquad j = 1,\ldots, \lambda.$$

On the other hand, since $\lim_{k\to\infty} p_k = q$ we have

$$q(x_i) = p_f(x_i) \qquad \text{for} \quad i = 1,\ldots, \mu \tag{5.5}$$

and

$$q'(y_j) = p_f'(y_j) \qquad \text{for} \quad j = 1,\ldots, \lambda. \tag{5.6}$$

Moreover, since both $q'(x) \geqslant 0$ on $[a, b]$ and $p_f'(x) \geqslant 0$ on $[a, b]$ we have

$$q''(y_j) = p_f''(y_j) \qquad \text{for all} \quad y_j \in (a, b). \tag{5.7}$$

Now by (1.6) the total number of conditions in (5.5), (5.6), and (5.7) is no less than $n + 2$. Thus, it follows as in [4] that the Birkhoff interpolation problem described by (5.5), (5.6), and (5.7) has a unique solution. Hence,

$$q \equiv p_f = T(f).$$

This is the desired contradiction, and Theorem 5.2, is proved. ∎

6. Remarks

We note here that in the counterexample in Section 2 the polynomials $p_\alpha(x)$ do not satisfy

$$p'_\alpha(x) \leqslant p'_f(x).$$

In fact,

$$p'_f(1) = 2(2 - 3^{1/2}) < 2(2 - 3^{1/2}) + 2\alpha + \alpha^2 = p'_\alpha(1).$$

Hence, as expected no p_α satisfies the hypotheses of Theorem 4.1.

On the other hand, to see how this example fits into the setting of Theorem 4.2 we observe that

$$\| p_f - p_\alpha \|' = | p_f(1) - p_\alpha(1)| = | p'_\alpha(1/3^{1/2})| = \alpha^2$$

and

$$\frac{\| f - p_\alpha \| - \| f - p_f \|}{\| p_f - p_\alpha \|'} = \frac{\alpha^2}{\alpha^2} = 1 \qquad \text{for all} \quad \alpha > 0.$$

The fact that strong unicity fails to hold for monotone approximation is somewhat surprising. On the other hand, the failure of classical theorems to hold for modified cases is not unusual and in fact an example is shown in Roulier and Taylor [7] which establishes that the polynomial of best approximation from a class of polynomials with restricted ranges of the first derivative need not in general be unique.

It would be interesting to investigate the other constrained approximation theories from this point of view. That is, for which problems does strong unicity hold.-

The question of whether or not the best monotone approximation operator satisfies a local Lipschitz condition remains open at this point.

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